

On the existence of Markovian superquadratic BSDEs with an unbounded terminal condition

Federica Masiero

Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca
via Cozzi 53, 20125 Milano, Italy
e-mail: federica.masiero@unimib.it

Adrien Richou

Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France.
CNRS, IMB, UMR 5251, F-33400 Talence, France.
INRIA, Équipe ALEA, F-33400 Talence, France.
e-mail: adrien.richou@math.u-bordeaux1.fr

July 2, 2012

Abstract

In [17], the author proved the existence and the uniqueness of solutions to Markovian superquadratic BSDEs with an unbounded terminal condition when the generator and the terminal condition are locally Lipschitz. In this paper, we prove that the existence result remains true for these BSDEs when the regularity assumptions on the generator and/or the terminal condition are weakened.

1 Introduction

Since the early nineties and the work of Pardoux and Peng [15], there has been an increasing interest for backward stochastic differential equations (BSDEs for short). These equations have a wide range of applications in stochastic control, in finance or in partial differential equation theory. A particular class of BSDE is studied since few years: BSDEs with generators of quadratic growth with respect to the variable z (quadratic BSDEs for short). This class arises, for example, in the context of utility optimization problems with exponential utility functions, or alternatively in questions related to risk minimization for the entropic risk measure (see e.g. [19, 11, 13] among many other references). Many papers deal with existence and uniqueness of solution for such BSDEs. In the first one [12], Kobylanski obtains an existence and uniqueness result for quadratic BSDEs when the terminal condition is bounded. Now, it is well known that the boundedness of the terminal condition is a too strong assumption. Indeed, when we look to the simple quadratic BSDE

$$Y_t = \xi + \int_t^T \frac{|Z_s|^2}{2} ds - \int_t^T Z_s dW_s,$$

we find the explicit solution $Y_t = \log(\mathbb{E}[e^\xi | \mathcal{F}_t])$ and we immediately see that we just need to have an exponential moment for ξ to obtain a solution. In [2], Briand and Hu show an existence result for quadratic BSDEs when the terminal condition has such an assumption. For the uniqueness problem, see e.g. [6].

Naturally, we could also wonder what happens when the generator has a superquadratic growth with respect to the variable z . Up to our knowledge the case of superquadratic BSDEs was firstly investigate in the recent paper [5]. In this article, authors consider superquadratic BSDEs when the terminal condition is bounded and the generator is convex in z . Firstly, they show that in a general way the problem is ill-posed: given a superquadratic generator, there exists a bounded terminal condition such that the associated BSDE does not admit any bounded solution and, on the other hand, if the BSDE admits a bounded solution, there exist infinitely many bounded solutions for this BSDE. In the same paper, authors also show that the problem becomes well-posed in a Markovian framework: When the terminal condition and the generator are deterministic functions of a forward SDE, we have

an existence result. More precisely, let us consider (X, Y, Z) the solution to the (decoupled) forward backward system

$$\begin{aligned} X_t &= x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s) dW_s, \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \end{aligned}$$

with growth assumptions

$$\begin{aligned} |f(t, x, y, z)| &\leq C(1 + |x|^{p_f} + |y| + |z|^{l+1}) \\ |g(x)| &\leq C(1 + |x|^{p_g}). \end{aligned}$$

In [5], authors obtain an existence result by assuming that $p_g = p_f = 0$, f is a convex function that depends only on z and g is a lower (or upper) semi-continuous function. As in the quadratic case it is possible to show that the boundedness of the terminal condition is a too strong assumption: in [17], author shows an existence and uniqueness result by assuming that $p_g \leq 1 + 1/l$, $p_f \leq 1 + 1/l$, f and g are locally Lipschitz functions with respect to x and z . When we consider this result, two questions arise:

- Could we have an existence result when p_g or p_f is upper than $1 + 1/l$?
- Could we have an existence result when f or g is less smooth with respect to x or z , that is to say, is it possible to have assumptions on the growth of g and f but not on the growth of their derivatives with respect to x and z ?

For the first question, the answer is clearly “no” in the quadratic case: see e.g. [6]. In the superquadratic case, authors of [10] have obtained the same limitation on the growth of the initial condition for the so-called generalized deterministic KPZ equation $u_t = u_{xx} + \lambda |u_x|^q$ and they show that this boundary is sharp for power-type initial conditions. So, it seems that the answer of the first question is also “no” in the superquadratic case.

For the second question, the answer is clearly “yes” in the quadratic case. Indeed, a smoothness assumption on f is required for uniqueness results (see e.g. [3, 6]) but not for existence results (see e.g. [3, 1]). In the superquadratic case, authors of [5] show an existence result when g is only lower (or upper) semi-continuous but also bounded. Nevertheless $f(z)$ is assume to be convex, that implies that it is a locally Lipschitz function. The aim of this paper is to obtain an existence result more general than [5], that is to say with g unbounded and f not necessarily locally Lipschitz with respect to z , and/or with functions less smooth than in [17]. More precisely, we have obtained two different existence results. For the first one, we assume that $(x, z) \mapsto f(t, x, y, z)$ and g are uniformly continuous functions with respect to a metric

$$d(x, x') = (1 + |x|^r + |x'|^r) |x - x'|, \quad (1.1)$$

with r well chosen. In this case, the smoothness of g is more restrictive than in the existence result of [5]. For the second existence result, we assume that the terminal condition is only upper semi-continuous.

For completeness, in the recent paper [4], Cheridito and Stadje show an existence and uniqueness result for superquadratic BSDEs in a “path-dependent” framework: the terminal condition and the generator are functions of Brownian motion paths. To the best of our knowledge, [5, 17, 4] are the only papers that deals with superquadratic BSDEs.

The paper is organized as follows. In section 2 we obtain some general a priori estimates on Y and Z for Markovian superquadratic BSDEs whereas sections 3 and 4 are devoted to the two different existence results described before.

Notations Throughout this paper, $(W_t)_{t \geq 0}$ will denote a d -dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geq 0$, let \mathcal{F}_t denotes the σ -algebra $\sigma(W_s; 0 \leq s \leq t)$, augmented with the \mathbb{P} -null sets of \mathcal{F} . The Euclidean norm on \mathbb{R}^d will be denoted by $|\cdot|$. The operator norm induced by $|\cdot|$ on the space of linear operators is also denoted by $|\cdot|$. The notation \mathbb{E}_t stands for the conditional expectation given \mathcal{F}_t . For $p \geq 2$, $m \in \mathbb{N}$, we denote further

- $\mathcal{S}^p(\mathbb{R}^m)$, or \mathcal{S}^p when no confusion is possible, the space of all adapted processes $(Y_t)_{t \in [0, T]}$ with values in \mathbb{R}^m normed by $\|Y\|_{\mathcal{S}^p} = \mathbb{E}[(\sup_{t \in [0, T]} |Y_t|)^p]^{1/p}$; $\mathcal{S}^\infty(\mathbb{R}^m)$, or \mathcal{S}^∞ , the space of bounded measurable processes;

- $\mathcal{M}^p(\mathbb{R}^m)$, or \mathcal{M}^p , the space of all progressively measurable processes $(Z_t)_{t \in [0, T]}$ with values in \mathbb{R}^m normed by $\|Z\|_{\mathcal{M}^p} = \mathbb{E}[(\int_0^T |Z_s|^2 ds)^{p/2}]^{1/p}$.

In the following, we keep the same notation C for all finite, nonnegative constants that appear in our computations.

In this paper we will consider X the solution to the SDE

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s) dW_s, \quad (1.2)$$

and $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$ the solution to the Markovian BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s. \quad (1.3)$$

2 Some a priori estimates on Y and Z

For the SDE (1.2) we use standard assumption.

Assumption (F.1). Let $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \rightarrow \mathbb{R}^{d \times d}$ be continuous functions and let us assume that there exists $K_b \geq 0$ such that:

- (a) $\forall t \in [0, T], |b(t, 0)| \leq C$,
- (b) $\forall t \in [0, T], \forall (x, x') \in \mathbb{R}^d \times \mathbb{R}^d, |b(t, x) - b(t, x')| \leq K_b |x - x'|$.

Now we will introduce some assumptions on the generator and the terminal condition of the BSDE (1.3).

Assumption (B.1). Let $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ be a continuous function and let us assume that there exist five constants, $l \geq 1, 0 \leq r_f < \frac{1}{l}, \beta \geq 0, \gamma \geq 0$ and $\delta \geq 0$ such that:

- (a) for each $(t, x, y, y', z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times d}$,

$$|f(t, x, y, z) - f(t, x, y', z)| \leq \delta |y - y'|;$$

- (b) for each $(t, x, y, z, z') \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$,

$$|f(t, x, y, z) - f(t, x, y, z')| \leq \left(C + \frac{\gamma}{2} (|z|^l + |z'|^l) \right) |z - z'|;$$

- (c) for each $(t, x, x', y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d}$,

$$|f(t, x, y, z) - f(t, x', y, z)| \leq \left(C + \frac{\beta}{2} (|x|^{r_f} + |x'|^{r_f}) \right) |x - x'|.$$

Assumption (TC.1). Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function and let us assume that there exist $0 \leq r_g < \frac{1}{l}$ and $\alpha \geq 0$ such that: for each $(t, x, x', y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d}$,

$$|g(x) - g(x')| \leq \left(C + \frac{\alpha}{2} (|x|^{r_g} + |x'|^{r_g}) \right) |x - x'|.$$

The aim of our work is to relax assumptions on local Lipschicity of functions to obtain following growth assumptions that are more natural for existence results.

Assumptions (B.2). Let $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ be a continuous function and let us assume that there exist five constants, $l \geq 1, 0 \leq p_f < 1 + \frac{1}{l}, \bar{\beta} \geq 0, \bar{\gamma} \geq 0, \bar{\delta} \geq 0$ such that: one of these inequalities holds, for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d}$,

- (a) $|f(t, x, y, z)| \leq C + \bar{\beta} |x|^{p_f} + \bar{\delta} |y| + \bar{\gamma} |z|^{l+1}$,
- (b) $-C - \bar{\beta} |x|^{p_f} - \bar{\delta} |y| - \bar{\gamma} |z| \leq f(t, x, y, z) \leq C + \bar{\beta} |x|^{p_f} + \bar{\delta} |y| + \bar{\gamma} |z|^{l+1}$.
- (c) $-C - \bar{\beta} |x|^{p_f} - \bar{\delta} |y| + \varepsilon |z|^{l+1} \leq f(t, x, y, z) \leq C + \bar{\beta} |x|^{p_f} + \bar{\delta} |y| + \bar{\gamma} |z|^{l+1}$.

Assumption (TC.2). Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a lower semi-continuous function and let us assume that there exist $0 \leq p_g < 1 + 1/l$ and $\bar{\alpha} \geq 0$ such that: for each $x \in \mathbb{R}^d$,

$$|g(x)| \leq C + \bar{\alpha} |x|^{p_g}.$$

Remark 2.1

- $(B.2)(c) \Rightarrow (B.2)(b) \Rightarrow (B.2)(a)$.
- $(B.1) \Rightarrow (B.2)(a)$ with $p_f = r_f + 1$.
- $(TC.1) \Rightarrow (TC.2)$ with $p_g = r_g + 1$.
- We will only consider quadratic and superquadratic BSDEs, so $l \geq 1$. The quadratic case corresponds to $l = 1$.

Firstly, let us recall the existence and uniqueness result shown in [17].

Proposition 2.2 We assume that assumptions (F.1), (B.1) and (TC.1) hold. There exists a solution (Y, Z) of the Markovian BSDE (1.3) in $\mathcal{S}^2 \times \mathcal{M}^2$ such that,

$$|Z_t| \leq A + B(|X_t|^{r_g} + (T-t)|X_t|^{r_f}), \quad \forall t \in [0, T].$$

Moreover, this solution is unique amongst solutions (Y, Z) such that

- $Y \in \mathcal{S}^2$,
- there exists $\eta > 0$ such that

$$\mathbb{E} \left[e^{(\frac{1}{2}+\eta)\frac{\gamma^2}{4} \int_0^T |Z_s|^{2l} ds} \right] < +\infty.$$

Remark 2.3 To be precise, in the article [17] the author shows the estimate

$$|Z_t| \leq A + B(|X_t|^{r_g \vee r_f}), \quad \forall t \in [0, T],$$

but it is rather easy to do the proof again to show the estimate given in Proposition 2.2.

Such a result allows us to obtain a comparison result.

Proposition 2.4 We assume that (F.1) holds. Let f_1, f_2 two generators and g_1, g_2 two terminal conditions such that (B.1) and (TC.1) hold. Let (Y^1, Z^1) and (Y^2, Z^2) be the associated solutions given by Proposition 2.2. We assume that $g_1 \leq g_2$ and $f_1 \leq f_2$. Then we have that $Y^1 \leq Y^2$ almost surely.

Proof of the proposition The proof is the same that the classical one that we can found in [7] for example. Let us set $\delta Y := Y^1 - Y^2$ and $\delta Z := Z^1 - Z^2$. The usual linearization trick gives us

$$\delta Y_t = g_1(X_T) - g_2(X_T) + \int_t^T f_1(s, X_s, Y_s^1, Z_s^1) - f_2(s, X_s, Y_s^1, Z_s^1) + \delta Y_s U_s + \delta Z_s V_s ds - \int_t^T \delta Z_s dW_s,$$

with $|U_s| \leq \delta$ and

$$|V_s| \leq C + \frac{\gamma}{2} \left(|Z_s^1|^l + |Z_s^2|^l \right) \leq C(1 + |X_s|^{(r_g \vee r_f)l}).$$

Since $(r_g \vee r_f)l < 1$, Novikov's condition is fulfilled and we are allowed to apply Girsanov's transformation:

$$\begin{aligned} \delta Y_t &= \mathbb{E}_t^{\mathbb{Q}} \left[e^{\int_t^T U_u du} (g_1(X_T) - g_2(X_T)) + \int_t^T e^{\int_t^s U_u du} (f_1(s, X_s, Y_s^1, Z_s^1) - f_2(s, X_s, Y_s^1, Z_s^1)) ds \right] \\ &\leq 0. \end{aligned}$$

□

Proposition 2.5 Let us assume that (F.1), (B.1), (B.2), (TC.1) and (TC.2) hold. Let (Y, Z) the solution of the BSDE (1.3) given by Proposition 2.2. Then we have, for all $t \in [0, T]$,

$$|Y_t| \leq C(1 + |X_t|^{p_g} + (T-t)|X_t|^{p_f})$$

with a constant C that depends on constants that appear in assumptions (F.1), (B.2) and (TC.2) but not in assumptions (B.1) and (TC.1).

Proof of the proposition Let us consider the terminal condition

$$\bar{g}(x) = C + \bar{\alpha}(|x| + 1)^{p_g},$$

and the generator

$$\bar{f}(t, x, y, z) = C + \bar{\beta}(|x| + 1)^{p_f} + \bar{\delta}|y| + \bar{\gamma}|z|^{l+1},$$

with C such that $g \leq \bar{g}$ and $f \leq \bar{f}$. (B.1) holds for \bar{f} and (TC.1) holds for \bar{g} , so, according to Proposition 2.2, there exists a unique solution (\bar{Y}, \bar{Z}) to the BSDE

$$\bar{Y}_t = \bar{g}(X_T) + \int_t^T \bar{f}(s, X_s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s dW_s.$$

Thanks to Proposition 2.4, we know that

$$Y \leq \bar{Y}, \quad \text{and} \quad \bar{Y} \geq 0.$$

Moreover, we have

$$\begin{aligned} \bar{Y}_t &\leq \mathbb{E}_t \left[e^{\bar{\delta}(T-t)} (C + \bar{\alpha}(|X_t| + 1)^{p_g}) + \int_t^T e^{\bar{\delta}(s-t)} (C + \bar{\beta}(|X_s| + 1)^{p_f} + \bar{\gamma}|\bar{Z}_s|^{l+1}) ds \right] \\ &\leq C \left(1 + \mathbb{E}_t \left[\sup_{t \leq s \leq T} |X_s|^{p_g} \right] + (T-t) \mathbb{E}_t \left[\sup_{t \leq s \leq T} |X_s|^{p_f} \right] \right), \end{aligned}$$

because $|\bar{Z}_s| \leq C(1 + |X_s|^{(p_g-1) \vee (p_f-1) \vee 0})$, $(p_g - 1)l < 1$ and $(p_f - 1)l < 1$. Let us remark that the constant C in the a priori estimate for \bar{Z} depends on constants that appear in assumptions (F.1), (B.2) and (TC.2) but not in assumptions (B.1) and (TC.1). Thanks to classical estimates on SDEs we have

$$\mathbb{E}_t \left[\sup_{t \leq s \leq T} |X_s|^p \right] \leq C(1 + |X_t|^p),$$

so we obtain

$$Y_t \leq \bar{Y}_t \leq C(1 + |X_t|^{p_g} + (T-t)|X_t|^{p_f}).$$

By the same type of argument we easily show that

$$-C(1 + |X_t|^{p_g} + (T-t)|X_t|^{p_f}) \leq Y_t.$$

□

Proposition 2.6 *Let us assume that (F.1), (B.1), (B.2), (TC.1) and (TC.2) hold. Let (Y, Z) the solution of the BSDE (1.3) given by Proposition 2.2. Then, for all $t \in [0, T]$,*

- *if we assume that (B.2)(b) holds, we have*

$$\mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right] \leq C(1 + |X_t|^{2p_g} + (T-t)|X_t|^{2p_f}),$$

- *if we assume that (B.2)(c) holds, we have*

$$\mathbb{E}_t \left[\int_t^T |Z_s|^{l+1} ds \right] \leq C(1 + |X_t|^{p_g} + (T-t)|X_t|^{p_f}),$$

with a constant C that depends on constants that appear in assumptions (F.1), (B.2) and (TC.2) but not in assumptions (B.1) and (TC.1).

Proof of the proposition Let us begin by giving some notations. We set, for $x \in \mathbb{R}^d$,

$$\varphi(t, x) := C(1 + |x|^{2p_g} + (T - t)^2 |x|^{2p_f})^{1/2},$$

with C such that $Y_t + \varphi(t, X_t) \geq 0$. This constant exists thanks to Proposition 2.5. Itô's formula gives us

$$\begin{aligned} d[Y_t + \varphi(t, X_t)] &= dY_t + {}^t \nabla_x \varphi(t, X_t) dX_t + \frac{1}{2} \text{tr} [\nabla_{xx}^2 \varphi(t, X_t) \sigma(t)^t \sigma(t)] dt + \partial_t \varphi(t, X_t) dt \\ &= \left[-f(t, X_t, Y_t, Z_t) + {}^t \nabla_x \varphi(t, X_t) b(t, X_t) + \frac{1}{2} \text{tr} [\nabla_{xx}^2 \varphi(t, X_t) \sigma(t)^t \sigma(t)] + \partial_t \varphi(t, X_t) \right] dt \\ &\quad + [{}^t \nabla_x \varphi(t, X_t) \sigma(t) + Z_t] dW_t \\ &:= A_t dt + B_t dW_t. \end{aligned}$$

For the first point, we will consider the process $P_t := (Y_t + \varphi(t, X_t))^2$. By applying Itô's formula and taking the conditional expectation we have

$$\mathbb{E}_t [P_t] + \mathbb{E}_t \left[\int_t^T |B_s|^2 ds \right] = \mathbb{E}_t [P_T] + 2\mathbb{E}_t \left[\int_t^T (Y_s + \varphi(s, X_s)) A_s ds \right].$$

By inequalities $|\nabla_x \varphi(t, x)| \leq C(1 + |x|^{p_f-1} + |x|^{p_g-1})$, $|\nabla_{xx}^2 \varphi(t, x)| \leq C$ and $|\partial_t \varphi(t, x)| \leq C(1 + |x|^{p_f} + |x|^{p_g})$, we show that

$$A_t \leq -f(t, X_t, Y_t, Z_t) + C(1 + |X_t|^{p_f} + |X_t|^{p_g}).$$

Then, we use inequalities $Y_t + \varphi(t, X_t) \geq 0$, (B.2)(b) and Proposition 2.5 to obtain

$$\begin{aligned} \mathbb{E}_t \left[\int_t^T (Y_s + \varphi(s, X_s)) A_s ds \right] &\leq \mathbb{E}_t \left[\int_t^T C(1 + |X_s|^{p_f} + |X_s|^{p_g})(1 + |X_s|^{p_f} + |X_s|^{p_g} + |Z_s|) ds \right] \\ &\leq \int_t^T C \left(1 + \mathbb{E}_t [|X_s|^{2p_g}] + \mathbb{E}_t [|X_s|^{2p_f}] \right) ds + \frac{1}{8} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right]. \end{aligned}$$

We have also, thanks to the growth assumption on g ,

$$P_T \leq C(1 + |X_T|^{2p_g}).$$

Finally, classical estimates on X give us

$$\mathbb{E}_t \left[\int_t^T |B_s|^2 ds \right] \leq C(1 + |X_t|^{2p_g} + (T - t) |X_t|^{2p_f}) + \frac{1}{4} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right],$$

and the result is proved because

$$\begin{aligned} \frac{1}{4} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right] &= \frac{1}{2} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right] - \frac{1}{4} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right] \\ &\leq \mathbb{E}_t \left[\int_t^T |B_s|^2 ds \right] + \mathbb{E}_t \left[\int_t^T |{}^t \nabla_x \varphi(s, X_s) \sigma(s)|^2 ds \right] - \frac{1}{4} \mathbb{E}_t \left[\int_t^T |Z_s|^2 ds \right] \\ &\leq C(1 + |X_t|^{2p_g} + (T - t) |X_t|^{2p_f}). \end{aligned}$$

For the second point, we just have to write

$$\begin{aligned} \mathbb{E}_t \left[\int_t^T |Z_s|^p ds \right] &\leq \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[\int_t^T f(s, X_s, Y_s, Z_s) ds \right] + \int_t^T C + \bar{\beta} |X_s|^{p_f} + \bar{\delta} |Y_s| ds \right) \\ &\leq \frac{1}{\varepsilon} \left(\mathbb{E}_t \left[Y_t - g(X_T) + \int_t^T C + \bar{\beta} |X_s|^{p_f} + \bar{\delta} |Y_s| ds \right] \right) \\ &\leq C(1 + (T - t) |X_t|^{p_f} + |X_t|^{p_g}) \end{aligned}$$

thanks to Proposition 2.5. □

Remark 2.7 Proposition 2.6 stays true if we replace assumption (B.2)(b) by

$$-C - \bar{\beta} |x|^{p_f} - \bar{\delta} |y| - \bar{\gamma} |z|^{l+1} \leq f(t, x, y, z) \leq C + \bar{\beta} |x|^{p_f} + \bar{\delta} |y| + \bar{\gamma} |z|,$$

and assumption (B.2)(c) by

$$-C - \bar{\beta} |x|^{p_f} - \bar{\delta} |y| - \bar{\gamma} |z|^{l+1} \leq f(t, x, y, z) \leq C + \bar{\beta} |x|^{p_f} + \bar{\delta} |y| - \varepsilon |z|^{l+1}.$$

3 A first existence result

Theorem 3.1 *Let assume that (F.1), (B.1)(a), (B.2)(c) and (TC.2) hold. We also assume that g and f are uniformly continuous functions with respect to x and z for the metric (1.1):*

$$\begin{aligned} \limsup_{\eta \rightarrow 0} \left(\sup \left\{ |g(x) - g(x')| \mid |x - x'| (1 + |x|^{(p_g-1) \vee 0} + |x'|^{(p_g-1) \vee 0}) < \eta \right\} \right) &= 0, \\ \limsup_{\eta \rightarrow 0} \left(\sup \left\{ |f(t, x, y, z) - f(t, x', y, z)| \mid \begin{array}{l} (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}, \\ |x - x'| (1 + |x|^{(p_f-1) \vee 0} + |x'|^{(p_f-1) \vee 0}) < \eta \end{array} \right\} \right) &= 0, \\ \limsup_{\eta \rightarrow 0} \left(\sup \left\{ |f(t, x, y, z) - f(t, x, y, z')| \mid \begin{array}{l} (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}, \\ |z - z'| (1 + |z|^l + |z'|^l) < \eta \end{array} \right\} \right) &= 0. \end{aligned}$$

Then, there exists a solution (Y, Z) to the BSDE (1.3) such that $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$. Moreover, we have

$$\mathbb{E} \left[\int_0^T |Z_s|^{l+1} ds \right] < +\infty.$$

Remark 3.2 *Standard examples of function uniformly continuous for the metric (1.1) are:*

- $f(t, x, y, z) := f_1(t, x, y, z) + f_2(t, x, y, z)$ such that (B.1) holds for f_1 and f_2 is continuous with respect to t , Lipschitz with respect to y and uniformly continuous with respect to x and z ,
- $g(x) := g_1(x) + g_2(x)$ such that (TC.1) holds for g_1 and g_2 is a uniformly continuous function.

Proof of the theorem Let us introduce an inf-convolution approximation of g and f : for $n \in \mathbb{N}$,

$$\begin{aligned} g_n(x) &:= \inf_{u \in \mathbb{R}^d} \left\{ g(u) + n |x - u| (1 + |x|^{(p_g-1) \vee 0} + |u|^{(p_g-1) \vee 0}) \right\}, \\ f_n(t, x, y, z) &:= \inf_{u \in \mathbb{R}^d, v \in \mathbb{R}^{1 \times d}} \left\{ f(t, u, y, v) + n |x - u| (1 + |x|^{(p_f-1) \vee 0} + |u|^{(p_f-1) \vee 0}) + n |z - v| (1 + |z|^l + |v|^l) \right\}. \end{aligned}$$

Let us recall some well-known facts about inf-convolution:

Lemma 3.3 *For $n \geq n_0$ with n_0 big enough, we have*

- g_n and f_n are well defined,
- (TC.1) holds for g_n with $r_g = (p_g - 1) \vee 0 < 1/l$,
- (TC.2) holds for g_n with C and $\bar{\alpha}$ that do not depend on n ,
- (B.1) holds for f_n with δ that does not depend on n and $r_f = (p_f - 1) \vee 0 < 1/l$,
- (B.2)(c) holds for f_n with $C, \varepsilon, \bar{\beta}, \bar{\delta}$ and $\bar{\gamma}$ that do not depend on n ,
- since f and g are uniformly continuous functions for the metric (1.1), we have

$$\|g - g_n\|_\infty + \|f - f_n\|_\infty \rightarrow 0.$$

Thanks to this lemma, we are able to apply Proposition 2.2: there exists a unique solution (Y^n, Z^n) to the BSDE (1.3) with the terminal condition g_n and the generator f_n . Let us set $\delta Y^{n,p} := Y^{n+p} - Y^n$ and $\delta Z^{n,p} := Z^{n+p} - Z^n$, for $n \geq n_0$ and $p \in \mathbb{N}$. As in the proof of Proposition 2.4, the usual linearization trick gives us

$$\begin{aligned} \delta Y_t^{n,p} &= g_{n+p}(X_T) - g_n(X_T) - \int_t^T \delta Z_s^{n,p} dW_s \\ &\quad + \int_t^T f_{n+p}(s, X_s, Y_s^{n+p}, Z_s^{n+p}) - f_n(s, X_s, Y_s^{n+p}, Z_s^{n+p}) + \delta Y_s^{n,p} U_s^{n,p} + \delta Z_s^{n,p} V_s^{n,p} ds. \end{aligned}$$

with $|U_s^{n,p}| \leq \delta$ and

$$|V_s^{n,p}| \leq C^{n,p} (1 + |Z_s^{n+p}|^l + |Z_s^n|^l) \leq C^{n,p} (1 + |X_s|^{((p_f-1) \vee (p_g-1) \vee 0)l}).$$

Since $((p_f - 1) \vee (p_g - 1) \vee 0)l < 1$, Novikov's condition is fulfilled and we are allowed to apply Girsanov's transformation:

$$\begin{aligned} \delta Y_t^{n,p} &= \mathbb{E}_t^{\mathbb{Q}^{n,p}} \left[e^{\int_t^T U_u^{n,p} du} (g_{n+p}(X_T) - g_n(X_T)) \right. \\ &\quad \left. + \int_t^T e^{\int_t^s U_u^{n,p} du} (f_{n+p}(s, X_s, Y_s^{n+p}, Z_s^{n+p}) - f_n(s, X_s, Y_s^{n+p}, Z_s^{n+p})) ds \right] \\ |\delta Y_t^{n,p}| &\leq \mathbb{E}_t^{\mathbb{Q}^{n,p}} \left[e^{\delta T} \|g_{n+p} - g_n\|_\infty + T e^{\delta T} \|f_{n+p} - f_n\|_\infty \right] \\ &\leq (1 + T) e^{\delta T} (\|g_{n+p} - g_n\|_\infty + \|f_{n+p} - f_n\|_\infty) \\ &\leq \varepsilon_n \end{aligned}$$

with ε_n a positive constant such that $\varepsilon_n \rightarrow 0$ when $n \rightarrow +\infty$. Itô's formula applied to $|\delta Y_t^{n,p}|^2$ and Burkholder-Davis-Gundy inequalities give us

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq t \leq T} |\delta Y_t^{n,p}|^2 \right] + \mathbb{E} \left[\int_0^T |\delta Z_t^{n,p}|^2 dt \right] \\ &\leq \mathbb{E} \left[|g_{n+p}(X_T) - g_n(X_T)|^2 \right] \\ &\quad + 2\mathbb{E} \left[\int_0^T |\delta Y_t^{n,p}| |f_{n+p}(s, X_s, Y_s^{n+p}, Z_s^{n+p}) - f_n(s, X_s, Y_s^n, Z_s^n)| ds \right] \\ &\quad + C\mathbb{E} \left[\left(\int_0^T |\delta Y_t^{n,p}|^2 |\delta Z_t^{n,p}|^2 dt \right)^{1/2} \right] \\ &\leq \|g_{n+p} - g_n\|_\infty^2 + C\varepsilon_n \mathbb{E} \left[\int_0^T |f_{n+p}(s, X_s, Y_s^{n+p}, Z_s^{n+p})| + |f_n(s, X_s, Y_s^n, Z_s^n)| ds \right] \\ &\quad + C\varepsilon_n \mathbb{E} \left[\left(\int_0^T |\delta Z_t^{n,p}|^2 dt \right) \right], \end{aligned}$$

with C that does not depend on n . Thanks to Lemma 3.3, we have

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq t \leq T} |\delta Y_t^{n,p}|^2 \right] + \mathbb{E} \left[\int_0^T |\delta Z_t^{n,p}|^2 dt \right] \\ &\leq \|g_{n+p} - g_n\|_\infty^2 \\ &\quad + C\varepsilon_n \left(1 + \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t|^{p_f} \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^{n+p}| \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n| \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T |Z_s^{n+p}|^{l+1} ds \right] + \mathbb{E} \left[\int_0^T |Z_s^n|^{l+1} ds \right] \right) \end{aligned}$$

with C that does not depend on n . Thanks to Lemma 3.3 we also know that we are allowed to apply Proposition 2.5 and Proposition 2.6 to (Y^{n+p}, Z^{n+p}) and (Y^n, Z^n) . So,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\delta Y_t^{n,p}|^2 \right] + \mathbb{E} \left[\int_0^T |\delta Z_t^{n,p}|^2 dt \right] \leq \|g_{n+p} - g_n\|_\infty^2 + C\varepsilon_n$$

with C that does not depend on n . Finally, $(Y^n, Z^n)_{n \geq n_0}$ is a Cauchy sequence in $\mathcal{S}^2 \times \mathcal{M}^2$: (Y^n, Z^n) tends to (Y, Z) in $\mathcal{S}^2 \times \mathcal{M}^2$ and Fatou's lemma gives us

$$\mathbb{E} \left[\int_0^T |Z_t|^{l+1} dt \right] \leq \liminf_{n \rightarrow +\infty} \mathbb{E} \left[\int_0^T |Z_t^n|^{l+1} dt \right] < +\infty.$$

Moreover, a simple application of the dominated convergence theorem show that (Y, Z) is a solution of the BSDE (1.3). \square

4 A second existence result

We will introduce new assumptions.

Assumption (F.2). b is differentiable with respect to x and σ is differentiable with respect to t . There exists $\lambda \in \mathbb{R}^+$ such that $\forall \eta \in \mathbb{R}^d$

$$\left| {}^t\eta\sigma(s)[{}^t\sigma(s){}^t\nabla b(s, x) - {}^t\sigma'(s)]\eta \right| \leq \lambda |{}^t\eta\sigma(s)|^2, \quad \forall (s, x) \in [0, T] \times \mathbb{R}^d.$$

Remark 4.1 It is shown in part 5.5.1 of [18] that if σ does not depend on time, assumption (F.2) is equivalent to this kind of commutativity assumption:

- there exist $A : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and $B : [0, T] \rightarrow \mathbb{R}^{d \times d}$ such that A is differentiable with respect to x , $\nabla_x A$ is bounded and $\forall x \in \mathbb{R}^d, \forall s \in [0, T], b(s, x)\sigma = \sigma A(s, x) + B(s)$.

It is also noticed in [18] that this assumption allows us to reduce assumption on the regularity of b by a standard smooth approximation of A .

Assumption (B.3). f is differentiable with respect to z and one of these inequalities is true, for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d}$,

- (a) $f(t, x, y, z) - \langle \nabla_z f(t, x, y, z), z \rangle \leq C + C|z|,$
- (b) $f(t, x, y, z) - \langle \nabla_z f(t, x, y, z), z \rangle \leq C - \varepsilon|z|^{l+1},$

Remark 4.2 Let us give some substancial examples of functions such that (B.3) holds. Let us assume that $f(t, x, y, z) := f_1(t, x, y, z) + f_2(t, x, y, z)$ is a differentiable function with respect to z .

- If f_1 is a Lipschitz function with respect to z and f_2 is a convex function with respect to z then (B.3)(a) holds.
- Let us assume that f_1 is a locally Lipschitz function with respect to z such that, $\exists p \in [0, l], \forall (t, x, y, z, z') \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$,

$$|f_1(t, x, y, z) - f_1(t, x, y, z')| \leq (1 + |z|^p + |z'|^p) |z - z'|,$$

and f_2 is a twice differentiable function with respect to z such that, $\forall (t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1 \times d}, \forall u \in \mathbb{R}^d,$

$${}^t u \nabla_{zz}^2 f_2(t, x, y, z) u \geq (-C + \varepsilon|z|^{l-1}) |u|^2.$$

Then we easily see that

$$f_1(t, x, y, z) - \langle \nabla_z f_1(t, x, y, z), z \rangle \leq C + C|z|^{p+1},$$

and a direct application of Taylor expansion with integral form gives us

$$f_2(t, x, y, z) - \langle \nabla_z f_2(t, x, y, z), z \rangle \leq C - C'|z|^{l+1},$$

so (B.3)(b) holds. For example, (B.3)(b) holds for the function $z \mapsto |z|^{l+1} + h(|z|^{l+1-\eta})$ with $\eta > 0$ and h a Lipschitz function.

Proposition 4.3 Let us assume that (F.1), (F.2), (B.1), (TC.1) and (TC.2) hold. Let (Y, Z) the solution of the BSDE (1.3) given by Proposition 2.2.

- If we assume that (B.3)(a) holds, $l < 2, 0 \leq p_g l < 1 - l/2$ and $p_f l < 1$, then we have, for all $t \in [0, T[$,

$$|Z_t| \leq \frac{C(1 + |X_t|^{p_g})}{\sqrt{T-t}} + C|X_t|^{p_f \vee r_f}.$$

- If we assume that (B.3)(b) holds, $0 \leq p_g l < 1$, then we have, for all $t \in [0, T[$,

$$|Z_t| \leq \frac{C(1 + |X_t|^{p_g/(l+1)})}{(T-t)^{1/(l+1)}} + C|X_t|^{\frac{p_f}{l+1} \vee r_f}.$$

The constant C depends on constants that appear in assumptions (F.1), (F.2), (B.1), (B.3) and (TC.2) but not in assumption (TC.1).

Proof of the proposition Firstly we will approximate our Markovian BSDE by another one. Let (Y^M, Z^M) the solution of the BSDE

$$Y_t^M = g_M(X_T) + \int_t^T f_M(s, X_s, Y_s^M, Z_s^M) ds - \int_t^T Z_s^M dW_s, \quad (4.1)$$

with $g_M = g \circ \rho_M$ and $f_M = f(\cdot, \rho_M(\cdot), \cdot, \cdot)$ where ρ_M is a smooth modification of the projection on the centered euclidean ball of radius M such that $|\rho_M| \leq M$, $|\nabla \rho_M| \leq 1$ and $\rho_M(x) = x$ when $|x| \leq M - 1$. It is now easy to see that g_M and f_M are Lipschitz functions with respect to x . Proposition 2.3 in [17] gives us that Z^M is bounded by a constant C_0 that depends on M . So, f_M is a Lipschitz function with respect to z and BSDE (4.1) is a classical Lipschitz BSDE. Now we will use the following Lemma that will be shown after.

Lemma 4.4 *Let us assume that (F.1), (F.2), (B.1), (TC.1) and (TC.2) hold.*

- *If we assume that (B.3)(a) holds, $l < 2$, $0 \leq p_g l < 1 - l/2$ and $p_f l < 1$, then we have, for all $t \in [0, T[$,*

$$|Z_t^M| \leq \frac{A_n + B_n |X_t|^{p_g}}{\sqrt{T-t}} + D_n |X_t|^{r_f \vee p_f},$$

with $(A_n, B_n, D_n)_{n \in \mathbb{N}}$ defined by recursion: $B_0 = 0$, $D_0 = 0$, $A_0 = C_0 \sqrt{T}$,

$$A_{n+1} = C(1 + A_n^{al} + B_n^{alp} + D_n^{al\bar{p}}), \quad B_{n+1} = C, \quad D_{n+1} = C,$$

where $a := p_g \vee p_f \vee r_f$, $p > 1$, $\bar{p} > 1$ and C is a constant that does not depend on M and constants in assumption (TC.1).

- *If we assume that (B.3)(b) holds, $0 \leq p_g l < 1$, then we have, for all $t \in [0, T[$,*

$$|Z_t^M| \leq \frac{A'_n + B'_n |X_t|^{p_g/(l+1)}}{(T-t)^{1/(l+1)}} + D'_n |X_t|^{\frac{p_f}{l+1} \vee r_f},$$

with $(A'_n, B'_n, D'_n)_{n \in \mathbb{N}}$ defined by recursion: $B'_0 = 0$, $D'_0 = 0$, $A'_0 = C_0 \sqrt{T}$,

$$A'_{n+1} = C(1 + A_n'^{a'l} + B_n'^{a'lp'} + D_n'^{a'l\bar{p}'}), \quad B'_{n+1} = C, \quad D'_{n+1} = C,$$

where $a' := (p_g \vee p_f \vee (l+1)r_f)/(l+1)$, $p' > 1$, $\bar{p}' > 1$ and C is a constant that does not depend on M and constants in assumption (TC.1).

Since $al < 1$ and $a'l < 1$, recursion functions that define sequences $(A_n)_{n \geq 0}$ and $(A'_n)_{n \geq 0}$ are contractor functions, so $A_n \rightarrow A_\infty$ and $A'_n \rightarrow A'_\infty$ when $n \rightarrow +\infty$, with A_∞ and A'_∞ that do not depend on M and constants in assumption (TC.1). Finally, we have:

- if we assume that (B.3)(a) hold, $l < 2$, $0 \leq p_g l < 1 - l/2$ and $p_f l < 1$, then we have, for all $t \in [0, T[$,

$$|Z_t^M| \leq \frac{C(1 + |X_t|^{p_g})}{\sqrt{T-t}} + C |X_t|^{p_f \vee r_f},$$

- if we assume that (B.3)(b) hold, $0 \leq p_g l < 1$, then we have, for all $t \in [0, T[$,

$$|Z_t^M| \leq \frac{C(1 + |X_t|^{p_g/(l+1)})}{(T-t)^{1/(l+1)}} + C |X_t|^{\frac{p_f}{l+1} \vee r_f}.$$

The constant C depends on constants that appear in assumptions (F.1), (F.2), (B.1), (B.3) and (TC.2) but not in assumption (TC.1). Moreover C does not depends on M . Now, we want to come back to the initial BSDE (1.3). It is already shown in the proof of Proposition 2.2 of the article [17] that $(Y^n, Z^n) \rightarrow (Y, Z)$ in $\mathcal{S}^2 \times \mathcal{M}^2$. So our estimates on Z^M stay true for a version of Z . \square

Proof of Lemma 4.4 Let us prove the result by recursion. For $n = 0$ we have already shown the result. Let us assume that the result is true for some $n \in \mathbb{N}$ and let us show that it stays true for $n + 1$. In a first time we will suppose that f and g are differentiable with respect to x and y . Then (Y^M, Z^M) is differentiable with respect to x and $(\nabla Y^M, \nabla Z^M)$ is the solution of the BSDE

$$\begin{aligned} \nabla Y_t^M &= \nabla g_M(X_T) \nabla X_T - \int_t^T \nabla Z_s^M dW_s \\ &\quad + \int_t^T \nabla_x f_M(s, X_s, Y_s^M, Z_s^M) \nabla X_s + \nabla_y f_M(s, X_s, Y_s^M, Z_s^M) \nabla Y_s^M + \nabla_z f_M(s, X_s, Y_s^M, Z_s^M) \nabla Z_s^M ds, \end{aligned}$$

and a version of Z^M is given by $(\nabla Y_t^M (\nabla X_t)^{-1} \sigma(t))_{t \in [0, T]}$. Let us introduce some notations: we set

$$\begin{aligned} d\tilde{W}_t &:= dW_t - \nabla_z f_M(t, X_t, Y_t^M, Z_t^M) dt, \\ \alpha_t &:= \int_0^t e^{\int_0^s \nabla_y f_M(u, X_u, Y_u^M, Z_u^M) du} \nabla_x f_M(s, X_s, Y_s^M, Z_s^M) \nabla X_s ds (\nabla X_t)^{-1} \sigma(t), \\ \tilde{Z}_t &:= e^{\int_0^t \nabla_y f_M(s, X_s, Y_s^M, Z_s^M) ds} Z_t^M + \alpha_t. \end{aligned}$$

By applying Girsanov's theorem we know that there exists a probability \mathbb{Q}^M under which \tilde{W} is a Brownian motion. Then, exactly as in the proof of Theorem 3.2 in [16], we can show that $|e^{\lambda t} \tilde{Z}_t|^2$ is a \mathbb{Q}^M -submartingale.

Firstly, we will show the first point of the lemma. Since $|e^{\lambda t} \tilde{Z}_t|^2$ is a \mathbb{Q}^M -submartingale, we have:

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T e^{2\lambda s} |\tilde{Z}_s^M|^2 ds \right] &\geq e^{2\lambda t} |\tilde{Z}_t^M|^2 (T - t) \\ &\geq e^{2\lambda t} \left| e^{\int_0^t \nabla_y f_M(s, X_s, Y_s^M, Z_s^M) ds} Z_t^M + \alpha_t \right|^2 (T - t), \end{aligned}$$

which implies

$$\begin{aligned} |Z_t^M|^2 (T - t) &\leq C \left(e^{2\lambda t} \left| e^{\int_0^t \nabla_y f_M(s, X_s, Y_s^M, Z_s^M) ds} Z_t^M + \alpha_t \right|^2 + |\alpha_t|^2 \right) (T - t) \\ &\leq C \left(\mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T e^{2\lambda s} |\tilde{Z}_s^M|^2 ds \right] + (T - t) (1 + |X_t|^{2r_f}) \right) \\ &\leq C \left(1 + \mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T |Z_s^M|^2 ds \right] + \mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T |X_s|^{2r_f} ds \right] + (T - t) |X_t|^{2r_f} \right). \quad (4.2) \end{aligned}$$

Let us recall that (Y^M, Z^M) is solution of BSDE

$$Y_t^M = g_M(X_T) + \int_t^T \tilde{f}_M(s, X_s, Y_s^M, Z_s^M) ds - \int_t^T Z_s^M d\tilde{W}_s,$$

with

$$\tilde{f}_M(s, x, y, z) := f_M(s, x, y, z) - \langle z, \nabla_z f_M(s, x, y, z) \rangle.$$

Since assumption (B.3)(a) holds for f , assumption (B.2)(b) holds for $-\tilde{f}_M$ with constants that do not depend on M . Then we can mimic the proof of the first point of Proposition 2.6 (see remark 2.7) to show that

$$\mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T |Z_s^M|^2 ds \right] \leq C \left(1 + \mathbb{E}_t^{\mathbb{Q}^M} [|X_T|^{2p_g}] + \int_t^T \mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^{2p_g}] + \mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^{2p_f}] ds \right), \quad (4.3)$$

with a constant C that does not depend on M and constants that appear in assumption (TC.1). Then, by putting (4.3) in (4.2), we see that we just have to obtain an a priori estimate for $\mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^c]$ with $c \in \mathbb{R}^{+*}$. We have

$$\begin{aligned} |X_s| &= \left| X_t + \int_t^s b(u, X_u) du + \int_t^s \sigma(u) d\tilde{W}_u + \int_t^s \sigma(u) \nabla_z f_M(u, X_u, Y_u^M, Z_u^M) du \right| \\ &\leq |X_t| + C + C \int_t^s |X_u| du + \left| \int_t^s \sigma(u) d\tilde{W}_u \right| + C \int_t^s |Z_u^M|^l du, \end{aligned}$$

with C that does not depend on M . Now we use the recursion assumption to obtain

$$\int_t^s |Z_u^M|^l du \leq C \int_t^s \frac{A_n^l}{(T-u)^{l/2}} + \frac{B_n^l}{(T-u)^{l/2}} |X_u|^{lp_g} + D_n^l |X_u|^{r_f l \vee p_f l} du.$$

Since $l/2 < 1$, $\int_t^T \frac{A_n^l}{(T-u)^{l/2}} du \leq C A_n^l$. For other terms we use Young inequality: Since $(r_f l \vee p_f l \vee p_g l) < 1$, we have

$$\int_t^s |Z_u^M|^l du \leq C A_n^l + C \int_t^s \frac{B_n^{lp}}{(T-u)^{lp/2}} + D_n^{l\bar{p}} + |X_u| du,$$

with $p = 1/(1-lp_g)$ and $\bar{p} > 1$. Since we assume that $lp_g < 1-l/2$, then $lp/2 < 1$ and $\int_t^s \frac{B_n^{lp}}{(T-u)^{lp/2}} du \leq C B_n^{lp}$. Finally, we obtain

$$\int_t^s |Z_u^M|^l du \leq C A_n^l + C B_n^{lp} + C D_n^{l\bar{p}} + C \int_t^s |X_u| du,$$

and

$$|X_s| \leq |X_t| + C + C \int_t^s |X_u| du + \left| \int_t^s \sigma(u) d\tilde{W}_u \right| + C A_n^l + C B_n^{lp} + C D_n^{l\bar{p}}.$$

Gronwall's lemma gives us

$$|X_s| \leq C \left(1 + \left| \int_t^s \sigma(u) d\tilde{W}_u \right| + A_n^l + B_n^{lp} + D_n^{l\bar{p}} + |X_t| \right)$$

that implies

$$\mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^c] \leq C (1 + A_n^{cl} + B_n^{clp} + D_n^{cl\bar{p}} + |X_t|^c). \quad (4.4)$$

By putting (4.4) in (4.3) and (4.2), we obtain

$$\begin{aligned} |Z_t^M|^2 (T-t) &\leq C \left(1 + \mathbb{E}_t^{\mathbb{Q}^M} [|X_T|^{2p_g}] + \int_t^T \mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^{2(p_g \vee p_f \vee r_f)}] ds + (T-t) |X_t|^{2r_f} \right) \\ &\leq C \left(1 + A_n^{2al} + B_n^{2alp} + D_n^{2al\bar{p}} + |X_t|^{2p_g} + (T-t) |X_t|^{2(p_f \vee r_f)} \right), \end{aligned}$$

with $a = p_g \vee p_f \vee r_f$ and C that does not depend on M and constants that appear in assumption (TC.1). So, we easily see that we can take

$$A_{n+1} = C(1 + A_n^{al} + B_n^{alp} + D_n^{al\bar{p}}), \quad B_{n+1} = C, \quad D_{n+1} = C,$$

and then the first point is proved.

To prove the second point, we will use the same machinery. Since $|e^{\lambda t} \tilde{Z}_t|^2$ is a \mathbb{Q}^M -submartingale, $|e^{\lambda t} \tilde{Z}_t|^{l+1}$ is also a \mathbb{Q}^M -submartingale. By the same calculus than previously, instead of having inequality (4.2), we obtain

$$|Z_t^M|^{l+1} (T-t) \leq C \left(1 + \mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T |Z_s^M|^{l+1} ds \right] + \mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T |X_s|^{(l+1)r_f} ds \right] + (T-t) |X_t|^{(l+1)r_f} \right). \quad (4.5)$$

Since assumption (B.3)(b) holds for f , assumption (B.2)(c) holds for $-\tilde{f}_M$ with constants that do not depend on M . Then we can mimic the proof of the second point of Proposition 2.6 (see remark 2.7) to show that

$$\mathbb{E}_t^{\mathbb{Q}^M} \left[\int_t^T |Z_s^M|^{l+1} ds \right] \leq C \left(1 + \mathbb{E}_t^{\mathbb{Q}^M} [|X_T|^{p_g}] + \int_t^T \mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^{p_g}] + \mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^{p_f}] ds \right), \quad (4.6)$$

with C that does not depend on M and constants that appear in assumption (TC.1). Then, by putting (4.6) in (4.5), we see that we just have to obtain an a priori estimate for $\mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^c]$ with $c \in \mathbb{R}^{+*}$. Once again, we have

$$|X_s| \leq |X_t| + C + C \int_t^s |X_u| du + \left| \int_t^s \sigma(u) d\tilde{W}_u \right| + C \int_t^s |Z_u^M|^l du,$$

with C that does not depend on M . Now we use the recursion assumption to obtain

$$\int_t^s |Z_u^M|^l du \leq C \int_t^s \frac{A_n^l}{(T-u)^{l/(l+1)}} + \frac{B_n^l}{(T-u)^{l/(l+1)}} |X_u|^{lp_g/(l+1)} + D_n^l |X_u|^{r_f l \vee p_f l/(l+1)} du.$$

Obviously we have $\int_t^T \frac{A_n^l}{(T-u)^{l/(l+1)}} du \leq C A_n^l$. For other terms we use Young inequality: Since $r_f l < 1$ and $(p_f l \vee p_g l)/(l+1) < 1$, we have

$$\int_t^s |Z_u^M|^l du \leq C A_n^l + C \int_t^s \frac{B_n^{lp}}{(T-u)^{lp/(l+1)}} + D_n^{l\bar{p}} + |X_u| du,$$

with $p = 1/(1-lp_g/(l+1))$ and $\bar{p} > 1$. Since we assume that $lp_g < 1$, then $lp/(l+1) < 1$ and $\int_t^s \frac{B_n^{lp}}{(T-u)^{lp/(l+1)}} du \leq C B_n^{lp}$. Finally, we obtain

$$\int_t^s |Z_u^M|^l du \leq C A_n^l + C B_n^{lp} + C D_n^{l\bar{p}} + C \int_t^s |X_u| du,$$

and

$$|X_s| \leq |X_t| + C + C \int_t^s |X_u| du + \left| \int_t^s \sigma(u) d\tilde{W}_u \right| + C A_n^l + C B_n^{lp} + C D_n^{l\bar{p}}.$$

Gronwall's lemma gives us

$$|X_s| \leq C \left(1 + \left| \int_t^s \sigma(u) d\tilde{W}_u \right| + A_n^l + B_n^{lp} + D_n^{l\bar{p}} + |X_t| \right)$$

that implies

$$\mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^c] \leq C (1 + A_n^{cl} + B_n^{clp} + D_n^{cl\bar{p}} + |X_t|^c). \quad (4.7)$$

By putting (4.7) in (4.6) and (4.5), we obtain

$$\begin{aligned} |Z_t^M|^{l+1} (T-t) &\leq C \left(1 + \mathbb{E}_t^{\mathbb{Q}^M} [|X_T|^{p_g}] + \int_t^T \mathbb{E}_t^{\mathbb{Q}^M} [|X_s|^{p_g \vee p_f \vee (l+1)r_f}] ds + (T-t) |X_t|^{(l+1)r_f} \right) \\ &\leq C \left(1 + A_n^{(l+1)a'l} + B_n^{(l+1)a'lp} + D_n^{(l+1)a'l\bar{p}} + |X_t|^{p_g} + (T-t) |X_t|^{p_f \vee (l+1)r_f} \right), \end{aligned}$$

with $a' = (p_g \vee p_f \vee (l+1)r_f)/(l+1)$. So, we easily see that we can take

$$A'_{n+1} = C(1 + A_n^{a'l} + B_n^{a'lp} + D_n^{a'l\bar{p}}), \quad B'_{n+1} = C, \quad D'_{n+1} = C,$$

and then the second point is proved.

When f and g are not differentiable we can prove the result by a standard approximation and stability results for BSDEs with linear growth. \square

Since estimates on Z given by Proposition 4.3 do not depend on constants that appear in assumption (TC.1), we can use it to show an existence result for superquadratic BSDEs with a quite general terminal condition.

Theorem 4.5 *Let assume that (F.1), (F.2), (B.1), (B.2)(b) and (TC.2) hold.*

- *If we assume that (B.3)(a) holds, $l < 2$, $0 \leq p_g l < 1 - l/2$ and $0 \leq p_f l < 1$, then there exists a solution (Y, Z) to the BSDE (1.3) such that $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$. Moreover we have, for all $t \in [0, T]$,*

$$|Z_t| \leq \frac{C(1 + |X_t|^{p_g})}{\sqrt{T-t}} + C |X_t|^{p_f \vee r_f}. \quad (4.8)$$

- *If we assume that (B.3)(b) holds and $0 \leq p_g l < 1$, then there exists a solution (Y, Z) to the BSDE (1.3) such that $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$. Moreover, we have for all $t \in [0, T]$,*

$$|Z_t| \leq \frac{C(1 + |X_t|^{p_g/(l+1)})}{(T-t)^{1/(l+1)}} + C |X_t|^{\frac{p_f}{l+1} \vee r_f}, \quad (4.9)$$

and, if we assume that (B.2)(c) holds,

$$\mathbb{E} \left[\int_0^T |Z_s|^{l+1} ds \right] < +\infty.$$

Proof of Theorem 4.5 The proof is based on the proof of Proposition 4.3 in [5]. For each integer $n \geq 0$, we construct the sup-convolution of g defined by

$$g_n(x) := \sup_{u \in \mathbb{R}^d} \{g(u) - n|x - u|\}.$$

Let us recall some well-known facts about sup-convolution:

Lemma 4.6 For $n \geq n_0$ with n_0 big enough, we have,

- g_n is well defined,
- (TC.1) holds for g_n with $r_g = 0$,
- (TC.2) holds for g_n with C and $\bar{\alpha}$ that do not depend on n ,
- $(g_n)_n$ is decreasing,
- $(g_n)_n$ converges pointwise to g .

Since (TC.1) holds, we can consider (Y^n, Z^n) the solution given by Proposition 2.2. It follows from Propositions 2.4 and 2.5 that

$$-C(1 + |X|^{p_f \vee p_g}) \leq Y^{n+1} \leq Y^n \leq Y^{n_0} \leq C(1 + |X|^{p_f \vee p_g}).$$

So $(Y_n)_n$ converges almost surely and we can define

$$Y = \lim_{n \rightarrow +\infty} Y^n.$$

It is easy to see that the estimate of Proposition 2.5 stays true for Y . Now the aim is to show that $(Z_n)_n$ converges in the good space. For any $T' \in]0, T[$, (Y^n, Z^n) satisfies

$$Y_t^n = Y_{T'}^n + \int_t^{T'} f(s, X_s, Y_s^n, Z_s^n) ds - \int_t^{T'} Z_s^n dW_s, \quad 0 \leq t \leq T'. \quad (4.10)$$

Let us denote $\delta Y^{n,m} := Y^n - Y^m$ and $\delta Z^{n,m} := Z^n - Z^m$. The classical linearization method gives us that $(\delta Y^{n,m}, \delta Z^{n,m})$ is the solution of BSDE

$$\delta Y_t^{n,m} = \delta Y_{T'}^{n,m} + \int_t^{T'} U_s^{n,m} \delta Y_s^{n,m} + V_s^{n,m} \delta Z_s^{n,m} ds - \int_t^{T'} \delta Z_s^{n,m} dW_s,$$

where $|U^{n,m}| \leq C$ and, by using estimates of Proposition 4.3,

$$|V^{n,m}| \leq C(1 + |Z^n|^l + |Z^m|^l) \leq C(1 + |X|^p), \quad (4.11)$$

with $p < 1$ and C that does not depends on n and m . Since $p < 1$, Novikov's condition is fulfilled and we can apply Girsanov's theorem: there exists a probability $\mathbb{Q}^{n,m}$ such that $d\tilde{W}_t := dW_t - V_t^{n,m} dt$ is a Brownian motion under this probability. By classical transformations, we have that $(\delta Y^{n,m}, \delta Z^{n,m})$ is the solution of the BSDE

$$\delta Y_t^{n,m} = \delta Y_{T'}^{n,m} e^{\int_t^{T'} U_s^{n,m} ds} - \int_t^{T'} e^{\int_t^s U_u^{n,m} du} \delta Z_s^{n,m} d\tilde{W}_s.$$

Since $U^{n,m}$ is bounded, classical estimates on BSDEs give us

$$\mathbb{E}^{\mathbb{Q}^{n,m}} \left[\left(\int_0^{T'} |\delta Z_s^{n,m}|^2 ds \right)^2 \right] \leq C \mathbb{E}^{\mathbb{Q}^{n,m}} \left[|\delta Y_{T'}^{n,m}|^4 \right]. \quad (4.12)$$

Now, we would like to have the same type of estimate than (4.12), but with the classical expectation. To do such a thing, we define the exponential martingale

$$\mathcal{E}_{T'}^{n,m} := \exp \left(\int_0^{T'} V_s^{n,m} dW_s - \frac{1}{2} \int_0^{T'} |V_s^{n,m}|^2 ds \right).$$

Then, for all $p \in \mathbb{R}$,

$$\mathbb{E} [(\mathcal{E}_{T'}^{n,m})^p] < C_p, \quad (4.13)$$

with C_p that does not depend on n and m : indeed, by applying (4.11) and Gronwall lemma we have

$$\begin{aligned} \mathbb{E} \left[e^{p \int_0^{T'} V_s^{n,m} dW_s - \frac{p}{2} \int_0^{T'} |V_s^{n,m}|^2 ds} \right] &= \mathbb{E} \left[e^{\frac{1}{2} \left(\int_0^{T'} 2p V_s^{n,m} dW_s - \frac{1}{2} \int_0^{T'} |2p V_s^{n,m}|^2 ds \right) + (p^2 - \frac{p}{2}) \int_0^{T'} |V_s^{n,m}|^2 ds} \right] \\ &\leq \mathbb{E} \left[e^{\int_0^{T'} 2p V_s^{n,m} dW_s - \frac{1}{2} \int_0^{T'} |2p V_s^{n,m}|^2 ds} \right] \mathbb{E} \left[e^{(2p^2 - p) \int_0^{T'} |V_s^{n,m}|^2 ds} \right] \\ &\leq \mathbb{E} \left[e^{C|2p^2 - p|(1 + \sup_{0 \leq s \leq T} |X_s|^{2p})} \right] \\ &< +\infty, \end{aligned}$$

because $2p < 2$. By applying Cauchy Schwarz inequality and by using (4.13) and (4.12), we obtain

$$\begin{aligned} \mathbb{E} \left[\int_0^{T'} |\delta Z_s^{n,m}|^2 ds \right] &= \mathbb{E} \left[(\mathcal{E}_{T'}^{n,m})^{-1/2} (\mathcal{E}_{T'}^{n,m})^{1/2} \int_0^{T'} |\delta Z_s^{n,m}|^2 ds \right] \\ &\leq \mathbb{E} \left[(\mathcal{E}_{T'}^{n,m})^{-1} \right]^{1/2} \mathbb{E}^{\mathbb{Q}^{n,m}} \left[\left(\int_0^{T'} |\delta Z_s^{n,m}|^2 ds \right)^2 \right]^{1/2} \\ &\leq C \mathbb{E}^{\mathbb{Q}^{n,m}} \left[|\delta Y_{T'}^{n,m}|^4 \right]^{1/2} \\ &\leq C \mathbb{E} \left[(\mathcal{E}_{T'}^{n,m})^2 \right]^{1/2} \mathbb{E} \left[|\delta Y_{T'}^{n,m}|^8 \right]^{1/4} \\ &\leq C \mathbb{E} \left[|\delta Y_{T'}^{n,m}|^8 \right]^{1/4} \xrightarrow{n,m \rightarrow 0} 0. \end{aligned}$$

Since \mathcal{M}^2 is a Banach space, we can define

$$Z = \lim_{n \rightarrow +\infty} Z^n, \quad \Omega \times [0, T[\text{-a.e..}$$

If we apply Proposition 2.6, we have that $\|Z^n\|_{\mathcal{M}^2} < C$ with a constant C that does not depend on n . So, Fatou's lemma gives us that $Z \in \mathcal{M}^2$. It is also easy to see that estimates on Z^n given by Proposition 4.3 stay true for Z . Moreover, if we assume that (B.2)(c) holds, then Proposition 2.6 gives us that

$$\mathbb{E} \left[\int_0^T |Z_s^n|^{l+1} ds \right] < C$$

with a constant C that does not depend on n and so

$$\mathbb{E} \left[\int_0^T |Z_s|^{l+1} ds \right] < C.$$

Finally, by passing to the limit when $n \rightarrow +\infty$ in (4.10) and by using the dominated convergence theorem, we obtain that for any fixed $T' \in [0, T[$, (Y, Z) satisfies

$$Y_t = Y_{T'} + \int_t^{T'} f(s, X_s, Y_s, Z_s) ds - \int_t^{T'} Z_s dW_s, \quad 0 \leq t \leq T'. \quad (4.14)$$

To conclude, we just have to prove that we can pass to the limit when $T' \rightarrow T$ in (4.14). Let us show that $Y_{T'} \xrightarrow{T' \rightarrow T} g(X_T)$ a.s.. Firstly, we have

$$\overline{\lim}_{s \rightarrow T} Y_s \leq \overline{\lim}_{s \rightarrow T} Y_s^n = g_n(X_T) \text{ a.s. for any } n \geq n_0,$$

which implies $\overline{\lim}_{s \rightarrow T} Y_s \leq g(X_T)$, a.s.. On the other hand, we use assumption (B.2)(b) and we apply Propositions 2.5 and 4.3 to deduce that, a.s.,

$$\begin{aligned} Y_t^n &= g_n(X_T) + \int_t^T f(s, X_s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s \\ &\geq g_n(X_T) - C \int_t^T 1 + |X_s|^{p_f} + |Y_s^n| + |Z_s^n| ds - \int_t^T Z_s^n dW_s \\ &\geq \mathbb{E}_t \left[g_n(X_T) - C \int_t^T 1 + |X_s|^{p_f \vee p_g \vee r_f} + \frac{|X_s|^{p_g}}{(T-s)^{1/2}} ds \right] \\ &\geq \mathbb{E}_t [g_n(X_T)] - C(T-t)(1 + |X_t|^{p_f \vee p_g \vee r_f}) - C\sqrt{T-t}(1 + |X_t|^{p_g}), \end{aligned}$$

and

$$Y_t = \lim_{n \rightarrow +\infty} Y_t^n \geq \mathbb{E}_t[g(X_T)] - C(T-t)(1 + |X_t|^{p_f \vee p_g \vee r_f}) - C\sqrt{T-t}(1 + |X_t|^{p_g}),$$

which implies

$$\lim_{t \rightarrow T} Y_t \geq \lim_{t \rightarrow T} \mathbb{E}_t[g(X_T)] = g(X_T).$$

Hence, $\lim_{t \rightarrow T} Y_t = g(X_T)$ a.s. .

Now, let us come back to BSDE (4.14). Since we have

$$\int_t^T f^-(s, X_s, Y_s, Z_s) ds \leq \int_t^T C(1 + |X_s|^{p_f} + |Y_s| + |Z_s|) ds < +\infty \text{ a.s.},$$

then

$$\int_t^{T'} f^-(s, X_s, Y_s, Z_s) ds \xrightarrow{T' \rightarrow T} \int_t^T f^-(s, X_s, Y_s, Z_s) ds < +\infty \text{ a.s.},$$

and

$$\begin{aligned} \int_t^{T'} f^+(s, X_s, Y_s, Z_s) ds &= Y_t - Y_{T'} + \int_t^{T'} f^-(s, X_s, Y_s, Z_s) ds + \int_t^{T'} Z_s dW_s \\ &\xrightarrow{T' \rightarrow T} Y_t - Y_T + \int_t^T f^-(s, X_s, Y_s, Z_s) ds + \int_t^T Z_s dW_s < +\infty \text{ a.s.} \end{aligned}$$

Finally, passing to the limit when $T' \rightarrow T$ in (4.14), we conclude that (Y, Z) is a solution to BSDE (1.3). \square

Remark 4.7 Estimate (4.8) is already known in the Lipschitz framework as a consequence of the Bismut-Elworthy formula (see e.g. [8]). For the superquadratic case, the same estimate was obtained when $p_g = p_f = 0$ in [5] (see also [16] for the quadratic case). In [5], Remark 4.4. gives the same type of estimate than (4.9) for the example $f(z) = |z|^l$. This result was already obtained by Gilding et al. in [9] using Bernstein's technique when $f(z) = |z|^l$, $b = 0$ and σ is the identity.

Remark 4.8 In this article, estimates (4.8) and (4.9) for the process Z allow us to obtain an existence result. But this type of deterministic bound is also interesting for numerical approximation of BSDEs (see e.g. [16]) or for studying stochastic optimal control problems in infinite dimension (see e.g. [14]).

References

- [1] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to general quadratic BSDEs and unbounded existence result. to appear in *Annals of Probability*.
- [2] P. Briand and Y. Hu. BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields*, **136**(4):604–618, 2006.
- [3] P. Briand and Y. Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probab. Theory Related Fields*, **141**(3-4):543–567, 2008.
- [4] P. Cheridito and M. Stadje. Existence, minimality and approximation of solutions to bsdes with convex drivers. *Stochastic Process. Appl.*, **122**(4):1540 – 1565, 2012.
- [5] F. Delbaen, Y. Hu, and X. Bao. Backward SDEs with superquadratic growth. *Probab. Theory Related Fields*, pages 1–48, 2010.
- [6] F. Delbaen, Y. Hu, and A. Richou. On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Ann. Inst. Henri Poincaré Probab. Stat.*, **47**(2):559–574, 2011.
- [7] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, **7**(1):1–71, 1997.
- [8] M. Fuhrman and G. Tessitore. The Bismut-Elworthy formula for backward SDEs and applications to nonlinear Kolmogorov equations and control in infinite dimensional spaces. *Stoch. Stoch. Rep.*, **74**(1-2):429–464, 2002.

- [9] B. H. Gilding, M. Guedda, and R. Kersner. The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$. *J. Math. Anal. Appl.*, 284(2):733–755, 2003.
- [10] A. Gladkov, M. Guedda, and R. Kersner. A KPZ growth model with possibly unbounded data: correctness and blow-up. *Nonlinear Anal.*, 68(7):2079–2091, 2008.
- [11] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *Ann. Appl. Probab.*, 15(3):1691–1712, 2005.
- [12] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.*, 28(2):558–602, 2000.
- [13] M. Mania and M. Schweizer. Dynamic exponential utility indifference valuation. *Ann. Appl. Probab.*, 15(3):2113–2143, 2005.
- [14] F. Masiero. Hamilton Jacobi Bellman equations in infinite dimensions with quadratic and superquadratic Hamiltonian. *Discrete Contin. Dyn. Syst.*, 32(1):223–263, 2012.
- [15] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.
- [16] A. Richou. Numerical simulation of BSDEs with drivers of quadratic growth. *Ann. Appl. Probab.*, 21(5):1933–1964, 2011.
- [17] A. Richou. Markovian quadratic and superquadratic BSDEs with an unbounded terminal condition. *Stochastic Process. Appl.*, 122(9):3173 – 3208, 2012.
- [18] A. Richou. *Étude théorique et numérique des équation différentielles stochastiques rétrogrades*. PhD thesis, Université de Rennes 1, November 2010.
- [19] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Math. Finance*, 10(2):259–276, 2000. INFORMS Applied Probability Conference (Ulm, 1999).